APPENDIX E

Riemann-Stieltjes Integrals

**Recall**: Consider the Riemann integral

\[
\int_a^b f(x) \, dx = \sum_{i=0}^{n-1} f(t_i)(x_{i+1} - x_i) \quad t_i \in [x_i, x_{i+1}].
\]

Consider the expectation introduced in Chapter 1,

\[
E[X] = \int_\Omega X \, dP = \int_{-\infty}^{\infty} x \, dF(x) = \int_{-\infty}^{\infty} xp(x) \, dx,
\]

(E.1)

where \( p \) is the probability density function of \( X \), and \( F \) is the cumulative distribution function of \( X \). The second integral in (E.1) is the Lebesgue integral, the fourth in (E.1) is the Riemann integral. What is the third integral in (E.1)?

**E.1. Definition**

**Basic Assumptions**: The functions \( f, g, \alpha, \beta \) are bounded on \( [a, b] \).

**Definition E.1**. Let \( P = \{x_1, x_2, \cdots, x_n\} \) be a partition of \( [a, b] \) and \( t_k \in [x_{k-1}, x_k] \) for \( k = 1, 2, \cdots, n \).

(1) A sum of the form

\[
S(P, f, \alpha) = \sum_{k=1}^{n} f(t_k)(\alpha(x_k) - \alpha(x_{k-1}))
\]

is called a Riemann-Stieltjes sum of \( f \) with respect to \( \alpha \).
(2) A function \( f \) is Riemann-Stieltjes Integrable with respect to \( \alpha \) on \([a, b]\), and we write “\( f \in R(\alpha) \) on \([a, b]\)”, if there exists \( A \in \mathbb{R} \) such that

\[
S(P, f, \alpha) \to A \quad \text{as} \quad \max_k |x_k - x_{k-1}| \to 0.
\]

也就是說分割地愈細，\( S(P, f, \alpha) \) 會愈接近 \( A \).

**Notation** E.2. If the number \( A \) exists in Definition E.1(2), it is uniquely determined and is denoted by

\[
\int_a^b f \, d\alpha \quad \text{or} \quad \int_a^b f(x) \, d\alpha(x).
\]

We also say that the Riemann-Stieltjes Integral \( \int_a^b f \, d\alpha \) exists.

**Example** E.3. Let \( f(x) = x \), and \( \alpha(x) = x + [x] \). Find \( \int_0^{10} f(x) \, d\alpha(x) \).

**Solution.** Consider the partition \( P = \left\{ 0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{10n}{n} \right\} \). Then

\[
S(P, f, \alpha) = \sum_{k=1}^{10n} f(t_k) \left( \alpha \left( \frac{k}{n} \right) - \alpha \left( \frac{k-1}{n} \right) \right)
= \sum_{k=1}^{10n} t_k \left( \left( \frac{k}{n} + \left[ \frac{k}{n} \right] \right) - \left( \frac{k-1}{n} + \left[ \frac{k-1}{n} \right] \right) \right)
= \sum_{k=1}^{10n} t_k \left( \frac{1}{n} + \left[ \frac{k}{n} \right] - \left[ \frac{k-1}{n} \right] \right)
= \sum_{k=1}^{10n} \frac{t_k}{n} + \sum_{k=1}^{10n} t_k \left( \left[ \frac{k}{n} \right] - \left[ \frac{k-1}{n} \right] \right).
\]

Since

\[
\sum_{k=1}^{10n} \frac{t_k}{n} \to \int_0^{10} x \, dx = \frac{x^2}{2}\bigg|_0^{10} = 50,
\]
and

\[
\sum_{k=1}^{10n} t_k \left( \left[ \frac{k}{n} \right] - \left[ \frac{k-1}{n} \right] \right) = \sum_{i=0}^{9} t_{(i+1)n}((i+1) - i) \to 55,
\]

where \( t_k \) are the points in the partition.

as \( n \to \infty \), we have

\[
\int_0^{10} f(x) \, d\alpha(x) = 50 + 55 = 105.
\]

E.2. Properties

**Theorem** E.4. Let \( c_1, c_2 \) be two constants in \( \mathbb{R} \).

1. If \( f, g \in R(\alpha) \) on \( [a, b] \), then \( c_1 f + c_2 g \in R(\alpha) \) on \( [a, b] \), and

\[
\int_a^b (c_1 f + c_2 g) \, d\alpha = c_1 \int_a^b f \, d\alpha + c_2 \int_a^b g \, d\alpha.
\]

2. If \( f \in R(\alpha) \) and \( f \in R(\beta) \) on \( [a, b] \), then \( f \in R(c_1 \alpha + c_2 \beta) \) on \( [a, b] \), and

\[
\int_a^b f \, d(c_1 \alpha + c_2 \beta) = c_1 \int_a^b f \, d\alpha + c_2 \int_a^b f \, d\beta.
\]

3. If \( c \in [a, b] \), then

\[
\int_a^b f \, d\alpha = \int_a^c f \, d\alpha + \int_c^b f \, d\alpha.
\]

**Definition** E.5. If \( a < b \), we define

\[
\int_b^a f \, d\alpha = -\int_a^b f \, d\alpha.
\]

**Theorem** E.6. If \( f \in R(\alpha) \) and \( \alpha \) has a continuous derivative on \( [a, b] \), then the Riemann integral \( \int_a^b f(x) \alpha'(x) \, dx \) exists and

\[
\int_a^b f(x) \, d\alpha(x) = \int_a^b f(x) \alpha'(x) \, dx.
\]
E.3. Technique of integrations

E.3.1. Integration by parts.

**Theorem** E.7 (Integration by parts). If \( f \in R(\alpha) \) on \([a, b]\), then \( \alpha \in R(f) \) on \([a, b]\), and
\[
\int_a^b f(x) \, d\alpha(x) = f(b)\alpha(b) - f(a)\alpha(a) - \int_a^b \alpha(x) \, df(x).
\]

**Example** E.8. As in Example E.3, \( f(x) = x \), and \( \alpha(x) = x + [x] \). Then
\[
\int_0^{10} f(x) \, d\alpha(x) = f(10)\alpha(10) - f(0)\alpha(0) - \int_0^{10} \alpha(x) \, df(x)
\]
\[
= 10 \times 20 - 0 \times 0 - \int_0^{10} (x + [x]) \, dx
\]
\[
= 200 - 50 - \int_0^{10} [x] \, dx = 150 - 45 = 105
\]

E.3.2. Change of variables.

**Theorem** E.9 (Change of variables). Suppose that \( f \in R(\alpha) \) on \([a, b]\) and \( g \) is a strictly increasing continuous function on \([c, d]\) with \( a = g(c) \), \( b = g(d) \). Let \( h = f \circ g \), \( \beta = \alpha \circ g \). Then \( h \in R(\beta) \) on \([c, d]\) and
\[
\int_a^b f(x) \, d\alpha(x) = \int_c^d f(g(t)) \, d\alpha(g(t)) = \int_c^d h(t) \, d\beta(t).
\]

**Example** E.10. Let \( y = \sqrt{x} \), we have
\[
\int_0^4 (\sqrt{x} + x^2) \, d\sqrt{x} = \int_0^2 (\sqrt{y} + y^4) \, dy = \int_0^2 \sqrt{y} \, dy + \int_0^2 y^4 \, dy
\]
\[
= 1 + \frac{1}{5} y^5 \bigg|_{y=0}^{y=2} = \frac{37}{5}
\]
E.3.3. **Step functions as** $\alpha$. By Remark C.6 and Theorem E.4(2), we have

$$
\int_a^b f(x) \, dF(x) = \int_a^b f(x) \, dF_{ac}(x) + \int_a^b f(x) \, dF_{sc}(x) + \int_a^b f(x) \, dF_d(x) \quad (E.2)
$$

其中 $\int_a^b f(x) \, dF_{ac}(x)$ 可利用 Theorem E.6 改成 Riemann integral. 在这一小节我们有兴趣的是讨论 $\int_a^b f(x) \, dF_d(x)$ 这个积分.

**Remark** E.11. If $\alpha \equiv \text{constant on } [a,b]$, then $S(P,f,\alpha) = 0$ for all partition $P$, and

$$
\int_a^b f(x) \, d\alpha(x) = 0.
$$

我们现有兴趣的是 $\alpha$ 为 step functions 时的积分.

**Theorem** E.12. Given $c \in (a,b)$. Define

$$
\alpha(x) = pI_{[a,c)} + rI_{\{c\}} + qI_{(c,b]}
$$

(ass given in Figure E.1). Suppose at least one of the functions $f$ or $\alpha$ is continuous from the left at $c$, and at least one is continuous from the right at $c$. Then $f \in R(\alpha)$ and

$$
\int_a^b f(x) \, d\alpha(x) = f(c)(\alpha(c+) - \alpha(c-)) = f(c)(q - p). \tag{1}
$$

**Remark** E.13. The integral $\int_a^b f \, d\alpha$ does not exist if both of $f$ and $\alpha$ are discontinuous from the left or from the right at $c$.

**Remark** E.14. (1) If $\alpha(x) = pI_{[a]} + qI_{(a,b]}$, then

$$
\int_a^b f(x) \, d\alpha(x) = f(a)(\alpha(a+) - \alpha(a))
$$

\[^1\text{Note that this value is independent of the value of } \alpha(c).\]
(2) If $\alpha(x) = pI_{[a,b]} + qI_{\{b\}}$, then

$$\int_a^b f(x)\,d\alpha(x) = f(b)(\alpha(b) - \alpha(b-))$$

**Example E.15.** (1) Consider

$$f(x) = 1 \quad \text{for} \quad x \in [-1,1], \quad \text{and} \quad \alpha(x) = -I_{\{0\}},$$

then

$$\int_{-1}^1 f(x)\,d\alpha(x) = f(0)(\alpha(0+) - \alpha(0-)) = 0$$

(2) Consider

$$f(x) = 2I_{\{0\}} + I_{[-1,0)\cup(0,1]} \quad \text{and} \quad \alpha(x) = -I_{[0,1]}.$$  

Then both of $\alpha$ and $f$ are discontinuous from the left at $x = 0$. This implies that the Riemann-Stieltjes integral $\int_{-1}^1 f\,d\alpha$ does not exist.
Theorem E.16 (Reduction of a Riemann-Stieltjes Integral to a finite sum). Let $\alpha$ be a step function on $[a, b]$ with jump

$$c_k = \alpha(x_k^+) - \alpha(x_k^-) \quad \text{at} \quad x = x_k.$$ 

Let $f$ be defined on $[a, b]$ in such a way that not both of $f$ and $\alpha$ are discontinuous from the left or from the right at $x_k$. Then $\int_a^b f(x) \, d\alpha(x)$ exists and

$$\int_a^b f(x) \, d\alpha(x) = \sum_{k=1}^n f(x_k)c_k.$$ 

Example E.17. (1) Let

$$f(x) = \begin{cases} 
3 & \text{if } x \leq 0 \\
3 - 4x & \text{if } 0 < x < 1 \\
-1 & \text{if } x \geq 1
\end{cases}$$

and

$$\alpha(x) = \begin{cases} 
0 & \text{if } x \leq 0 \\
2 & \text{if } 0 < x < 1 \\
0 & \text{if } x \geq 1
\end{cases}$$

Since $f$ is continuous, $\int_{-3}^3 f(x) \, d\alpha(x)$ exists and

$$\int_{-3}^3 f(x) \, d\alpha(x) = f(0)(\alpha(0^+) - \alpha(0^-)) + f(1)(\alpha(1^+) - \alpha(1^-))$$

$$= 3(2 - 0) + (-1)(0 - 2) = 8.$$ 

(2) Let $\alpha(x) = 2I_{[0,2]} + 5I_{[2,3]} + 6I_{[3,\infty)}$

$$\int_{-5}^{10} e^{-3x} \, d\alpha(x) = e^{-3 \cdot 0}(2 - 0) + e^{-3 \cdot 2}(5 - 2) + e^{-3 \cdot 3}(6 - 5)$$

$$= 2 + 3e^{-6} + e^{-9}.$$
在這節的最後，我們看看一個 $\int_a^b f(x) \, dF_{sc}(x)$ 的例子。

**Example** E.18. Suppose $F$ is the Cantor function (see Figure C.1). By integration by parts, we have

$$
\int_0^1 x \, dF(x) = xF(x)|_{x=0}^1 - \int_0^1 F(x) \, dx = 1 - \int_0^1 F(x) \, dx.
$$

Since $\int_0^1 F(x) \, dx$ is the area of the Cantor function on $[0,1]$, we get

$$
\int_0^1 F(x) \, dx = \frac{1}{2}.
$$

Hence,

$$
\int_0^1 x \, dF(x) = \frac{1}{2}.
$$

### E.3.4. Comparison theorem

**Theorem** E.19. Assume that $\alpha$ is an increasing function on $[a, b]$. If $f, g \in R(\alpha)$ on $[a, b]$, and if $f(x) \leq g(x)$ for $x \in [a, b]$, then

$$
\int_a^b f(x) \, d\alpha(x) \leq \int_a^b g(x) \, d\alpha(x).
$$

**Corollary** E.20. If $g(x) \geq 0$ and $\alpha$ is an increasing function on $[a, b]$, then

$$
\int_a^b f(x) \, d\alpha(x) \geq 0.
$$

**Theorem** E.21. Assume that $\alpha$ is an increasing function on $[a, b]$. If $f \in R(\alpha)$ on $[a, b]$, then

1. $|f| \in R(\alpha)$ on $[a, b]$, and

$$
\left| \int_a^b f(x) \, d\alpha(x) \right| \leq \int_a^b |f(x)| \, d\alpha(x).
$$

2. $f^2 \in R(\alpha)$ on $[a, b]$. 
Theorem E.22. Assume that $\alpha$ be an increasing function on $[a, b]$. If $f, g \in R(\alpha)$ on $[a, b]$, then $f \cdot g \in R(\alpha)$.

E.4. Bounded variation and Riemann-Stieltjes integral

Definition E.23. A function $\alpha : [a, b] \rightarrow \mathbb{R}$ is said to be of bounded variation if there exists a constant $M$ such that

$$\sum_{k=1}^{n} |\alpha(x_k) - \alpha(x_{k-1})| \leq M$$

for every partition $\{x_0, x_1, \ldots, x_n\}$ of $[a, b]$.

Bounded variation 說穿了就是講數上下震盪總和為 bounded. 但哪些函數會是 of bounded variation?

Theorem E.24. Let $\alpha$ be defined on $[a, b]$, then $\alpha$ is of bounded variation on $[a, b]$, if and only if there exist two increasing functions $\alpha_1$ and $\alpha_2$, such that $\alpha = \alpha_1 - \alpha_2$.

Theorem E.25. If $f$ is continuous on $[a, b]$, and if $\alpha$ is of bounded variation on $[a, b]$, then $f \in R(\alpha)$. Moreover, the function

$$F(t) = \int_{0}^{t} f(x) \, d\alpha(x)$$

has the following properties:

1. $F$ is of bounded variation on $[a, b]$.
2. Every continuous point of $\alpha$ is also a continuous point of $F$. 