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On Linear Preservers of lgw-Majorization on $\mathbf{M}_{n,m}$

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Abstract. Let $\mathbf{M}_{n,m}$ be the set of all $n\times m$ matrices with entries in $\mathbb{F}$, where $\mathbb{F}$ is the field of real or complex numbers. For $A,B\in\mathbf{M}_{n,m}$, we say that B is lgw-majorized (left generalized weakly majorized) by A if there exists an $n\times n$ g-row stochastic (generalized row stochastic) matrix R such that $B=RA$. In this paper, we characterize all linear operators that strongly preserve lgw-majorization on $\mathbf{M}_{n,m}$ and all linear operators that strongly preserve left weak matrix majorization on $\mathbf{M}_n$.

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1. Introduction

Suppose that $\mathbf{M}_n := \mathbf{M}_{n,n}$. A matrix $R\in\mathbf{M}_n$ is a generalized row stochastic matrix (g-row stochastic, for short) if $Re=e$, where $e=(1,1,...,1)^t$, see [8]. Recall that R is row stochastic if it has nonnegative entries and $Re=e$. Given $A,B\in\mathbf{M}_{n,m}$, B is said to be left (respectively right) weakly matrix majorized by A, and write $A\succ_{lw} B$ (respectively $A\succ_{rw} B$) if there exists a row stochastic matrix R such that $B=RA$ (respectively $B=AR$), see [9, 12].

A linear operator $T:\mathbf{M}_{n,m}\to\mathbf{M}_{n,m}$ preserves an order relation $\succ$ in $\mathbf{M}_{n,m}$, if $T(X)\succ T(Y)$ whenever $X\succ Y$. Also, T is said to strongly preserve $\succ$ if

$$X\succ Y \Longleftrightarrow T(X)\succ T(Y).$$

In [7], Beasley, Lee and Lee proved that if a linear operator $T:\mathbf{M}_n\to\mathbf{M}_n$ strongly preserves right weak matrix majorization, then there exist a permutation P and an invertible matrix $M\in\mathbf{M}_n$ such that $T(X)=MXP$ for every X in $\mathrm{span}(\mathbf{R}_n)$, where $\mathbf{R}_n$ is the set of all $n\times n$ row stochastic matrices. Recently Hasani and Radjabalipour in [10] showed that

$$T(X)=MXP, \quad \forall\, X\in\ \mathbf{M}_n.$$

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$T : \mathbf{M}_n \rightarrow \mathbf{M}_n$ strongly preserves left weak matrix majorization, then there exist a permutation P and an invertible matrix $M \in \mathbf{M}_n$ such that

$$T(X) = PXM, \quad \forall X \in \mathbf{M}_n. \tag{1.1}$$

Definition 1.1. *Let $A, B \in \mathbf{M}_{n,m}$. The matrix B is said to be lgw-majorized by A if there exists an $n \times n$ g-row stochastic matrix R such that $B = RA$ (denoted $A \succ_{lgw} B$). Analogously, B is said to be rgw-majorized (right generalized weakly majorized) by A (denoted $A \succ_{rgw} B$) if there exists an $m \times m$ g-row stochastic matrix R such that $B = AR$.*

The notions of rgw and lgw-majorization were motivated by the concepts of right and left weak matrix majorization which were introduced in [9] and [12] respectively. In [5] the authors introduced the notion of lgw-majorization on $\mathbf{M}_n$ and characterized its strong linear preservers. Also, in [1], all strong linear preservers of rgw-majorization on $M_{n,m}$ were characterized. We would like to point out that there is no duality between the cases of rgw and lgw-majorization and that the proofs are essentially different. In this paper, we will show that a linear operator $T : \mathbf{M}_{n,m} \rightarrow \mathbf{M}_{n,m}$ strongly preserves lgw-majorization if and only if there exist invertible matrices A and B such that A is g-row stochastic and $T(X) = AXB$ for every X in $\mathbf{M}_{n,m}$. Also, we prove that the relation (1.1) (one of the theorems in [10]) may be obtained as a corollary of our Proposition 2.1. For more information on the type of majorization and linear preservers of majorization see [2, 3, 4, 11].

Throughout this paper, $\mathbf{GR}_n$ is the set of all $n \times n$ g-row stochastic matrices, $e = (1, \ldots, 1)^t \in \mathbb{F}^n$ and $\mathbf{J} = ee^t \in \mathbf{M}_n$.

2. lgw-Majorization

In this section, we state some properties of lgw-majorization on $\mathbf{M}_{n,m}$. Also we characterize all linear operators on $\mathbf{M}_{n,m}$ that strongly preserve lgw-majorization. First we state some lemmas.

Lemma 2.1. *Let $T : \mathbf{M}_{n,m} \rightarrow \mathbf{M}_{n,m}$ be a linear operator that strongly preserves lgw-majorization. Then T is invertible.*

Proof. Suppose $T(A) = 0$. Notice that since T is linear, we have $T(0) = 0 = T(A)$. Then it is obvious that $T(0) \succ_{lgw} T(A)$. Therefore, $0 \succ_{lgw} A$ because T strongly preserves lgw-majorization. Then, there exists an $n \times n$ g-row stochastic matrix R such that $A = R0$. So, $A = 0$, and hence T is invertible. ∎

The set of g-row stochastic matrices and the lgw-majorization relation on $\mathbf{M}_{n,m}$ has the following properties. Proofs are not given.

Remark 2.1. Let A and B be two g-row stochastic matrices . Then AB and A^{-1} (if A is invertible) are g-row stochastic matrices too.

Remark 2.2. Let $X, Y \in \mathbf{M}_{n,m}$, $A, B \in \mathbf{GR}_n$, $C \in \mathbf{M}_m$, and $\alpha, \beta \in \mathbb{F}$ such that A, B, and C are invertible and $\alpha \neq 0$. Then the following conditions are equivalent.

1. $X \succ_{lgw} Y$.
2. $AX \succ_{lgw} BY$.
3. $\alpha X + \beta \mathbf{J}_{n,m} \succ_{lgw} \alpha Y + \beta\, \mathbf{J}_{n,m}$.
4. $XC \succ_{lgw} YC$.

Here $\mathbf{J}_{n,m}$ is the $n \times m$ matrix all of whose entries are equal to one.

Now, we characterize the linear operators preserving lgw-majorization on $\mathbb{F}^n$.

Lemma 2.2. *Let $x \in \mathbb{F}^n$. Then $x \succ_{lgw} y$ for every $y \in \mathbb{F}^n$ if and only if $x \notin \operatorname{span}\{e\}$.*

Proof. If $x \succ_{lgw} y$ for every $y \in \mathbb{F}^n$, it is clear that $x \notin \operatorname{span}\{e\}$. Conversely, let $x = (x_1, \dots, x_n)^t \notin \operatorname{span}\{e\}$. Then x has at least two distinct components such as x_k and x_l. Let $y = (y_1, \dots, y_n)^t$ be an arbitrary vector in $\mathbb{F}^n$. For every i, j $(1 \le i, j \le n)$, put $r_{ik} = (y_i - x_l)/(x_k - x_l)$, $r_{il} = (-y_i + x_k)/(x_k - x_l)$, and $r_{ij} = 0$ if $j \neq k, l$. It is easy to show that, $R = [r_{ij}] \in \mathbf{GR}_n$ and $Rx = y$. Then $x \succ_{lgw} y$. ∎

Lemma 2.3. *A nonzero linear operator $T : \mathbb{F}^n \to \mathbb{F}^n$ preserves lgw-majorization if and only if $x \notin \operatorname{span}\{e\}$ implies that $T(x) \notin \operatorname{span}\{e\}$.*

Proof. Let T preserve lgw-majorization. Suppose that $x \notin \operatorname{span}\{e\}$, then $x \succ_{lgw} y$ for every $y \in \mathbb{F}^n$ by Lemma 2.2. Therefore $T(x) \succ_{lgw} T(y)$ for every $y \in \mathbb{F}^n$. Assume, if possible, $T(x) \in \operatorname{span}\{e\}$, then $T(y) = T(x)$ for every $y \in \mathbb{F}^n$, and hence $T = 0$, which is a contradiction. Then, $T(x) \notin \operatorname{span}\{e\}$. Conversely, letting $x \notin \operatorname{span}\{e\}$ implies that $T(x) \notin \operatorname{span}\{e\}$. Assume that $x \succ_{lgw} y$. If $x \in \operatorname{span}\{e\}$, then $x = y$ and hence $T(x) = T(y)$. If $x \notin \operatorname{span}\{e\}$, then $T(x) \notin \operatorname{span}\{e\}$ by hypothesis. So $T(x) \succ_{lgw} z$ for every $z \in \mathbb{F}^n$ by Lemma 2.2, and hence $T(x) \succ_{lgw} T(y)$. Then T preserves lgw-majorization. ∎

Theorem 2.1. *A linear operator $T : \mathbb{F}^n \to \mathbb{F}^n$ preserves lgw-majorization if and only if one of the following assertions holds.*

(i) *There exists $R \in \mathbf{M}_n$ such that $\ker(R) = \operatorname{span}\{e\}$, $e \notin \operatorname{Im}(\mathrm{R})$, and $T(x) = Rx$ for every $x \in \mathbb{F}^n$.*

(ii) *There exist an invertible matrix $R \in \mathbf{GR}_n$ and $\alpha \in \mathbb{F}$ such that $T(x) = \alpha Rx$ for every $x \in \mathbb{F}^n$.*

Proof. If T satisfies (i) or (ii), it is easy to show that T preserves lgw-majorization. Conversely, let T preserve lgw-majorization. If $T = 0$, we may choose $\alpha = 0$. So suppose that $T \neq 0$. Let A be the matrix representation of T with respect to the canonical basis of $\mathbb{F}^n$. If T is invertible, then there exists $b \in \mathbb{F}^n$ such that $Ab = e$. So $b = re$, for some nonzero $r \in \mathbb{F}$, by Lemma 2.3. Then $Ae = \frac{1}{r}e$ and hence $T(x) = \alpha Rx$, where $\alpha = \frac{1}{r}$ and $R = (rA) \in \mathbf{GR}_n$ is invertible. If T is singular, then by Lemma 2.3, $\ker(T) = \operatorname{span}\{e\}$ and $e \notin \operatorname{Im}(T)$. So $\ker(A) = \operatorname{span}\{e\}$ and $e \notin \operatorname{Im}(A)$. ∎

Corollary 2.1. *If $T : \mathbb{F}^n \to \mathbb{F}^n$ is a nonzero linear preserver of lgw-majorization, then $\operatorname{rank}(T)$ is equal to n or $n-1$.*

Proof. By Theorem 2.1, $\ker(T) = \{0\}$ or $\ker(T) = \operatorname{span}\{e\}$. Then $\operatorname{rank}(T) = n$ or $\operatorname{rank}(T) = n-1$. ∎

Now, we state the following two lemmas to prove the main theorem of this paper.

Lemma 2.4. *For every invertible matrix $A \in \mathbf{GR}_n$, the following assertions are true.*

(i) *If $AR = RA$ for every g-row stochastic matrix R, then $A = I$.*

(ii) *If $(x + Ay) \succ_{lgw} (Rx + ARy)$ for every $R \in \mathbf{GR}_n$ and every $x, y \in \mathbb{F}^n$, then $A = I$.*

Proof. (i) For every i $(1 \le i \le n)$ assume that R_i is the matrix with e as i^{th} column and 0 elsewhere. Then $R_i \in \mathbf{GR}_n$. Since $A \in \mathbf{GR}_n$ is invertible and $AR_i = R_iA$ for every i $(1 \le i \le n)$, it is easy to see that $A = I$.

(ii) Observe that since A is invertible, condition (ii) can be rewritten as follows:

$$x+y \succ_{lgw} Rx+ARA^{-1}y \ , \ \forall\, R \in \mathbf{GR}_n \ , \ \forall\, x,y \in \mathbb{F}^n.$$

Put $x = e-e_i$ and $y = e_i$ in the above relation, where $\{e_1,\ldots,e_n\}$ is the canonical basis of $\mathbb{F}^n$. Thus, $e \succ_{lgw} [e-(R-ARA^{-1})e_i]$ for every i $(1 \le i \le n)$. So $(R-ARA^{-1})e_i = 0$ for every i $(1 \le i \le n)$. Therefore, $RA = AR$ for every $R \in \mathbf{GR}_n$, and hence $A = I$ by part (i). ∎

Lemma 2.5. *Let $A \in \boldsymbol{M}_n$. If* $\ker(A) = \operatorname{span}\{e\}$*, then there exist some $x_0, y_0 \in \mathbb{F}^n$ and $R_0 \in$ $\mathbf{GR}_n$ such that $R_0x_0 + AR_0y_0$ is not lgw-majorized by $x_0 + Ay_0$.*

Proof. Assume if possible,

$$x+Ay \succ_{lgw} Rx+ARy, \forall x,y \in \mathbb{F}^n, \forall R \in \mathbf{GR}_n \ . \tag{2.1}$$

If $e \in \operatorname{Im}(A)$, then there exists $y_0 \in \mathbb{F}^n$ such that $Ay_0 = e$. Put $x = 0$ and $y = y_0$ in (2.1). So $e = Ay_0 \succ_{lgw} ARy_0$, and hence $ARy_0 = e$ for every $R \in \mathbf{GR}_n$. Then $Ay = e$ for every $y \in \mathbb{F}^n$, which is a contradiction. If $e \notin \operatorname{Im}(A)$, then $\mathbb{F}^n = \operatorname{Im}(A) \bigoplus \operatorname{span}\{e\}$. So for every i $(1 \le i \le n)$, there exist $y_i \in \mathbb{F}^n$ and $r_i \in \mathbb{F}$ such that $e_i = Ay_i + r_ie$, where e_i is the i^{th} vector in the canonical basis of $\mathbb{F}^n$. Put $x = e-(e_i - r_ie)$ and $y = y_i$ in (2.1). Then

$$r_ie - Re_i + ARy_i = 0, \ \forall R \in \mathbf{GR}_n. \tag{2.2}$$

For every j $(1 \le j \le n, j \ne i)$, put $R_j = ee_j^t$ in (2.2). Then $r_i = 0$ for every i $(1 \le i \le n)$. Therefore, $Ay_i = e_i$ for every i $(1 \le i \le n)$. It thus follows that $\operatorname{Im}(A) = \mathbb{F}^n$, which is a contradiction . ∎

Remark 2.3. Let $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ be a linear operator. For every i,j $(1 \le i,j \le m)$, consider the embedding $E^j : \mathbb{F}^n \to \mathbf{M}_{n,m}$ and the projection $E_i : \mathbf{M}_{n,m} \to \mathbb{F}^n$ which are defined by $E^j(x) = xe_j^t$ and $E_i(X) = Xe_i$, respectively. Put $T_i^j = E_iTE^j$ for every i,j $(1 \le i,j \le m)$. If $X = [x_1|\ldots|x_m] \in \mathbf{M}_{n,m}$, where x_i is the i^{th} column of X, then

$$T(X) = T([x_1|\ldots|x_m]) = \left[\sum_{j=1}^{m} T_1^j(x_j)|\ldots|\sum_{j=1}^{m} T_m^j(x_j)\right].$$

Moreover, if T preserves lgw-majorization, then for every i,j $(1 \le i,j \le m)$ T_i^j preserves lgw-majorization too.

Now, we state the main theorem of this section.

Theorem 2.2. *Let $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ be a linear operator. Then T strongly preserves lgw-majorization if and only if $T(X) = AXB$ for every $X \in \mathbf{M}_{n,m}$, where $A \in \mathbf{GR}_n$ and $B \in \boldsymbol{M}_m$ are invertible .*

Proof. If $m = 1$, the result is proved by Theorem 2.1. So we may suppose that $m \ge 2$. As the sufficiency of the condition is easy to see, only we prove the necessity of the condition. Suppose that T strongly preserves lgw-majorization. Since T_i^j preserves lgw-majorization for every i,j $(1 \le i,j \le m)$, then, by Theorem 2.1, there exist $\alpha_i^j \in \mathbb{F}$ and $A_i^j \in \mathbf{M}_n$ such that $T_i^j(x) = \alpha_i^jA_i^jx$, where either $A_i^j \in \mathbf{GR}_n$ is invertible or $\ker(A_i^j) = \operatorname{span}\{e\}$ and $e \notin \operatorname{Im}(A_i^j)$. Then

$$T(X) = \left[\sum_{j=1}^{m} \alpha_i^jA_i^jx_j|\ldots|\sum_{j=1}^{m} \alpha_m^jA_m^jx_j\right]. \tag{2.3}$$

Now, we consider three steps for the proof.

Step 1. In this step, we will show that if there exist p and q $(1 \le p,q \le m)$ such that $\alpha_p^q \neq 0$ and $A_p^q \in \mathbf{GR}_n$ is invertible, then for every j $(1 \le j \le m)$, $A_p^j = A_p^q$.

If $\alpha_p^j = 0$, without loss of generality, we can choose $A_p^j = A_p^q$. Suppose that $\alpha_p^j \neq 0$. For every $x,y \in \mathbb{F}^n$, put $X = xe_q^t + ye_j^t$. Then $T(X) \succ_{lgw} T(RX)$ for all $R \in \mathbf{GR}_n$, and hence by (2.3)

$$\begin{aligned}
&\alpha_p^q A_p^q x + \alpha_p^j A_p^j y \succ_{lgw} \alpha_p^q A_p^q Rx + \alpha_p^j A_p^j Ry, \forall x,y \in \mathbb{F}^n, \forall R \in \mathbf{GR}_n \\
&\Rightarrow x + (A_p^q)^{-1} A_p^j (\frac{\alpha_p^j}{\alpha_p^q} y) \succ_{lgw} Rx + (A_p^q)^{-1} A_p^j R(\frac{\alpha_p^j}{\alpha_p^q} y), \forall x,y \in \mathbb{F}^n, \forall R \in \mathbf{GR}_n \\
&\Rightarrow x + (A_p^q)^{-1} A_p^j y \succ_{lgw} Rx + (A_p^q)^{-1} A_p^j Ry, \forall x,y \in \mathbb{F}^n, \forall R \in \mathbf{GR}_n.
\end{aligned}$$

So by Lemma 2.5, A_p^j is invertible, and hence, by Lemma 2.4, $A_p^j = A_p^q$. Set $A_p = A_p^q$. Then

$$T(X) = \left[\sum_{j=1}^m \alpha_1^j A_1^j x_j | \dots | A_p \sum_{j=1}^m \alpha_p^j x_j | \dots | \sum_{j=1}^m \alpha_m^j A_m^j x_j \right].$$

Step 2. In this step we will show that for every i and j $(1 \le i,j \le m)$, $A_i^j \in \mathbf{GR}_n$ is invertible if $\alpha_i^j \neq 0$. Assume if possible there exist r and s $(1 \le r,s \le m)$ such that $ker(A_r^s) = \text{span}\{e\}$ and $\alpha_r^s \neq 0$. Without loss of generality, we can assume that $r = m$. Then by step 1, for every j $(1 \le j \le m)$ we obtain $\ker(A_m^j) = \text{span}\{e\}$. Now, we construct a nonzero $n \times m$ matrix U such that $T(U) = 0$. Consider the vectors

$$b_1 = \begin{pmatrix} \alpha_1^1 \\ \vdots \\ \alpha_{m-1}^1 \end{pmatrix}, \dots, b_m = \begin{pmatrix} \alpha_1^m \\ \vdots \\ \alpha_{m-1}^m \end{pmatrix} \in \mathbb{F}^{m-1}.$$

It is clear that $\{b_1, \dots, b_m\}$ is a linearly dependent set in $\mathbb{F}^{m-1}$. So there exist (not all zero) $\lambda_1, \dots, \lambda_m \in \mathbb{F}$ such that

$$\sum_{j=1}^m \lambda_j \alpha_i^j = 0 \text{ , } \forall\, i \in \{1, \dots, m-1\}.$$

Now, define $U := [\lambda_1 e | \dots | \lambda_m e] \in \mathbf{M}_{n,m}$. It is clear that, $U \neq 0$ and

$$T(U) = \left[\sum_{j=1}^m \lambda_j \alpha_1^j A_1^j e | \dots | \sum_{j=1}^m \lambda_j \alpha_m^j A_m^j e \right].$$

We will show that $T(U) = 0$. Since $\ker(A_m^j) = \text{span}\{e\}$, it is clear that $\sum_{j=1}^m \lambda_j \alpha_m^j A_m^j e$ =0 and hence the last column of $T(U)$ is zero. Now, for every k $(1 \le k \le m-1)$, we consider the k^{th} column of $T(U)$.

Case 1. Let $\alpha_k^l \neq 0$ and $A_k^l \in \mathbf{GR}_n$ be invertible for some l $(1 \le l \le m)$. Then, by step 1

$$\sum_{j=1}^m \lambda_j \alpha_k^j A_k^j e = A_k^l (\sum_{j=1}^m \lambda_j \alpha_k^j) e = 0.$$

So the k^{th} column of $T(U)$ is 0 .

Case 2. Suppose that $\alpha_k^j A_k^j$ is non invertible for every j $(1 \leq j \leq m)$. Then $\text{span}\{e\} \subseteq \ker(\alpha_k^j A_k^j)$, by Theorem 2.1. So $\sum_{j=1}^m \lambda_j \alpha_k^j A_k^j e = 0$, and hence the k^{th} column of $T(U)$ is 0. Therefore $T(U) = 0$. But by Lemma 2.1, we know that T is invertible and hence a contradiction is obtained. So for every i and j $(1 \leq i, j \leq m)$, $A_i^j \in \mathbf{GR}_n$ is invertible if $\alpha_i^j \neq 0$. Then, by step 1, there exist invertible matrices $A_i \in \mathbf{GR}_n$ $(1 \leq i \leq m)$ such that $T(X) = T[x_1|\ldots|x_m] = [A_1 X a_1|\ldots|A_m X a_m]$, where $a_i = (\alpha_i^1, \ldots, \alpha_i^m)^t$ for every i $(1 \leq i \leq m)$.

Step 3. In this step, we will show that $A_i = A_1$ for every i $(1 \leq i \leq m)$. First, we show that $\text{rank}[a_1|\ldots|a_m] \geq 2$. Assume, if possible, $\{a_1, \ldots, a_m\} \subseteq \text{span}\{a\}$ for some $a \in \mathbb{F}^m$. Since $m \geq 2$, then we may choose a nonzero vector $b \in (\text{span}\{a\})^\perp$. Define $X_0 := e_1 b^t \in \mathbf{M}_{n,m}$. It is clear that $X_0 \neq 0$ and $T(X_0) = 0$, which is a contradiction and hence $\text{rank}[a_1|\ldots|a_m] \geq 2$. Without loss of generality, we can assume that $\{a_1, a_2\}$ is a linearly independent set. Let $X \in \mathbf{M}_{n,m}$ and $R \in \mathbf{GR}_n$ be arbitrary. Then

$$\begin{aligned} X \succ_{lgw} RX &\Rightarrow T(X) \succ_{lgw} T(RX) \\ &\Rightarrow [A_1 X a_1|\ldots|A_m X a_m] \succ_{lgw} [A_1 R X a_1|\ldots|A_m R X a_m] \\ &\Rightarrow A_1 X a_1 + A_2 X a_2 \succ_{lgw} A_1 R X a_1 + A_2 R X a_2 \\ &\Rightarrow X a_1 + (A_1^{-1} A_2) X a_2 \succ_{lgw} R X a_1 + (A_1^{-1} A_2) R X a_2. \end{aligned} \tag{2.4}$$

Since $\{a_1, a_2\}$ is linearly independent, for every $x, y \in \mathbb{F}^n$, there exists $B_{x,y} \in \mathbf{M}_{n,m}$ such that $B_{x,y}\, a_1 = x$ and $B_{x,y}\, a_2 = y$. Putting $X = B_{x,y}$ in (2.4), we see that

$$\begin{aligned} B_{x,y}\, a_1 + (A_1^{-1}\, A_2)\, B_{x,y}\, a_2 \succ_{lgw} R B_{x,y}\, a_2 + (A_1^{-1}\, A_2)\, R B_{x,y}\, a_2 \Rightarrow \\ x + (A_1^{-1} A_2)\, y \succ_{lgw} Rx + (A_1^{-1} A_2) Ry\ , \forall R \in \mathbf{GR}_n. \end{aligned}$$

Then, by Lemma 2.4, $A_1^{-1} A_2 = I$, and hence $A_2 = A_1$. Since T is invertible, it is easy to see that $a_i \neq 0$ for every i $(3 \leq i \leq m)$. Consequently $\{a_1, a_i\}$ or $\{a_2, a_i\}$ is a linearly independent set. By a similar argument as in the above, we conclude that $A_i = A_1$ or $A_i = A_2$. Let $A = A_1$. It follows that $A_i = A$ for every i $(1 \leq i \leq m)$. Therefore,

$$T(X) = [AXa_1\ |\ldots|AXa_m] = AXB,$$

where $B = [a_1|\ldots|\ a_m]$ is an invertible matrix in $\mathbf{M}_m$. ∎

The following statements show that every strong linear preserver of left weak matrix majorization is a strong linear preserver of lgw-majorization but the converse is false.

Lemma 2.6. *For every g-row stochastic matrix $R \in \mathbf{GR}_n$, there exist row stochastic matrices $R_1, \ldots, R_4 \in \boldsymbol{M}_n(\mathbb{R})$ and scalars $r_1, \ldots, r_4 \in \mathbb{C}$ such that $\sum_{i=1}^4 r_i = 1$ and $R = \sum_{i=1}^4 r_i R_i$.*

Proof. Let $R = A + iB$, where A and B are real $n \times n$ matrices. Since we know that $Re = e$, we obtain that $Ae = e$ and $Be = 0$. Assume that $A = [a_{i,j}]$ and $B = [b_{i,j}]$. Put $\alpha = \max\{0, -a_{i,j} : 1 \leq i, j \leq n\}$ and $\beta = \max\{-b_{i,j} : 1 \leq i, j \leq n\}$. Define $R_1 := (1/(1+n\alpha))(A + \alpha\mathbf{J})$ and $R_2 = R_3 := (1/n)\mathbf{J}$. Also, $R_4 := (1/(n\beta))(B + \beta\mathbf{J})$ if $\beta \neq 0$, and $R_4 := (1/n)\mathbf{J}$ if $\beta = 0$. It is clear that $R_1, \ldots, R_4$ are row stochastic matrices and

$$R = A + iB = (1 + n\alpha)R_1 + (-n\alpha)R_2 + (in\beta)R_3 + (-in\beta)R_4. \qquad ∎$$

Proposition 2.1. *Let $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ be a linear operator that strongly preserves left weak matrix majorization. Then T strongly preserves lgw-majorization.*

Proof. Let $A \succ_{lgw} B$. Then there exists a g-row stochastic matrix R such that $B = RA$. For the g-row stochastic matrix R, there exist scalars $r_1, \ldots, r_4$ and row stochastic matrices $R_1, \ldots, R_4$ such that $\sum_{i=1}^{4} r_i = 1$ and $R = \sum_{i=1}^{4} r_i R_i$ by Lemma 2.6. For every i $(1 \leq i \leq 4)$, $A \succ_{lw} R_i A$ and hence $T(A) \succ_{lw} T(R_i A)$. Then there exist row stochastic matrices S_i $(1 \leq i \leq 4)$, such that $T(R_i A) = S_i T(A)$. Put $S = \sum_{i=1}^{4} r_i S_i$. It is clear that S is a g-row stochastic matrix and $T(B) = ST(A)$. Therefore, $T(A) \succ_{lgw} T(B)$. On the other hand, replacing T by T^{-1}, in a similar fashion we conclude that $A \succ_{lgw} B$ whenever $T(A) \succ_{lgw} T(B)$. Then T strongly preserves lgw-majorization. ∎

Example 2.1. Let the linear operator $T : \mathbf{M}_2 \to \mathbf{M}_2$ be such that $T(X) = AX$ for every $X \in \mathbf{M}_2$, where

$$A = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}.$$

It is clear that T strongly preserves lgw-majorization by Theorem 2.2. But T does not preserve left weak matrix majorization because

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \succ_{lw} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \not\succ_{lw} T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Now, we state the following corollary which characterizes all linear operators that strongly preserve left weak matrix majorization on $\mathbf{M}_n$.

Remark 2.4. Let A be an invertible row stochastic matrix. If A^{-1} is row stochastic matrix, then A is a permutation.

Corollary 2.2. [10, Theorem 5.2] *A linear operator $T : \mathbf{M}_n \to \mathbf{M}_n$ strongly preserves left weak matrix majorization $\succ_{lw}$ if and only if $T(X) = PXL$, where P is permutation and $L \in \boldsymbol{M}_n$ is invertible.*

Proof. Suppose that T strongly preserves left weak matrix majorization. Then T strongly preserves lgw-majorization by Proposition 2.1. Therefore in view of Theorem 2.2, there exist invertible matrices $A \in \mathbf{GR}_n$ and $B \in \mathbf{M}_n$ such that $T(X) = AXB$ for all $X \in \mathbf{M}_n$. For every row stochastic matrix R, it is clear that $I \succ_{lw} R$. So $T(I) \succ_{lw} T(R)$ for every row stochastic matrix R. Consequently $AIB \succ_{lw} ARB$, and hence ARA^{-1} is a row stochastic matrix for every row stochastic matrix R. It is easy to show that A^{-1} is a row stochastic matrix. Similarly, A is a row stochastic matrix too, and hence A is a permutation matrix. ∎

3. Rank-1-preserver and lgw-majorization

In this section, we are using the structure of rank-1-preservers on $\mathbf{M}_{n,m}$ to study the strong linear preservers of lgw-majorization. This comment has been suggested by one of the anonymous referees. We recall that a rank-k-preserver is a linear operator T on $\mathbf{M}_{n,m}$ such that $\operatorname{rank}(T(A)) = k$ whenever $\operatorname{rank}(A) = k$. The following notations are fixed through this section.

For every $A \in \mathbf{M}_{n,m}$, the notation $\mathscr{R}(A)$ is denoted for the set of all rows of A. The symbol $\mathbb{F}_m$ is used for the set of all $1 \times m$ row vectors with entries in $\mathbb{F}$. Let $E \subseteq \mathbb{F}_m$, the cardinal number of E is denoted by $|E|$ and the g-convex hull of E is the following set:

$$\text{g-conv}(E)=\left\{\sum_{i=1}^{n} r_i x_i : r_i\in\mathbb{F}, x_i\in E\ , n\in\mathbb{N}, \sum_{i=1}^{n} r_i=1\right\}.$$

Let $A,B\in\mathbf{M}_{n,m}$. One can easily show that $A\succ_{lgw} B$ if and only if $\mathscr{R}(B)\subset$ g-conv$(\mathscr{R}(A))$.

Proposition 3.1. *For every $A\in\mathbf{M}_{n,m}$, the following assertions are true.*

(i) $\{B\in\mathbf{M}_{n,m} : A\succ_{lgw} B\}=\{A\}$ *if and only if* $|\mathscr{R}(A)|=1$.

(ii) $\{B\in\mathbf{M}_{n,m} : A\succ_{lgw} B\}$ *is a subspace of* $\mathbf{M}_{n,m}$ *with dimension n if and only if* rank$(A)=1$ *and* $|\mathscr{R}(A)|>1$.

Proof. It is easy to show that (i) holds, so just we prove (ii). Assume that rank$(A)=1$ and $|\mathscr{R}(A)|>1$, then there exist a nonzero $x\in\mathbb{F}_m$ and scalars $r_1,\dots,r_n\in\mathbb{F}$ (not all equal) such that $A=(r_1x,\dots,r_nx)^t$ Then $\mathscr{R}(A)=\{r_1x,\dots,r_nx\}$ and there exist $r_i,r_j\in\{r_1,\dots,r_n\}$ such that $r_i\neq r_j$. Since $r_i\neq r_j$, we have g-conv$(\{r_ix,r_jx\})=\{\alpha x:\alpha\in\mathbb{F}\}$. Therefore

$$\begin{aligned}\{B\in\mathbf{M}_{n,m} : A\succ_{lgw} B\} &= \{B\in\mathbf{M}_{n,m} : \mathscr{R}(B)\subset \text{g-conv}(\mathscr{R}(A))\}\\ &= \{B\in\mathbf{M}_{n,m} : \mathscr{R}(B)\subset\{\alpha x:\alpha\in\mathbb{F}\}\}\\ &= \text{span}\left\{\begin{pmatrix}x\\0\\\vdots\\0\end{pmatrix},\dots,\begin{pmatrix}0\\\vdots\\0\\x\end{pmatrix}\right\},\end{aligned}$$

which is a subspace of $\mathbf{M}_{n,m}$ with dimension n. Conversely, if rank$(A)>1$, then there exist $x,y\in\mathbb{F}_m$ such that $\{x,y\}\subset\mathscr{R}(A)$ and $\{x,y\}$ is a linearly independent set. If $W=\{B\in\mathbf{M}_{n,m} : A\succ_{lgw} B\}$ is a subspace of $\mathbf{M}_{n,m}$, then it is clear that $0\in$ g-conv$\{\mathscr{R}(A)\}$ and hence

$$\left\{\begin{pmatrix}x\\0\\\vdots\\0\end{pmatrix},\dots,\begin{pmatrix}0\\\vdots\\0\\x\end{pmatrix},\begin{pmatrix}y\\0\\\vdots\\0\end{pmatrix}\right\}$$

is a linearly independent set in W. Thus $\dim W\geq n+1$. ∎

Proposition 3.2. *Let $T:\mathbf{M}_{n,m}\to\mathbf{M}_{n,m}$ be a linear operator. If T strongly preserves $\succ_{lgw}$, then T is a rank-1-preserver.*

Proof. Assume that $A\in\mathbf{M}_{n,m}$ and rank$(A)=1$. Now, we consider two cases for the proof.

Case 1. Suppose that $|\mathscr{R}(A)|=1$. Then by part (i) of Proposition 3.1, $\{B\in\mathbf{M}_{n,m} : A\succ_{lgw} B\}=\{A\}$. Since T is invertible, we have $T(A)\neq 0$ and hence rank$(T(A))\geq 1$. Assume if possible rank$(T(A))>1$. By part (i) of Proposition 3.1 and invertibility of T, there exists $B\in\mathbf{M}_{n,m}$ such that $T(A)\succ_{lgw} T(B)$ and $T(B)\neq T(A)$. Since T strongly preserves $\succ_{lgw}$, we have $A\succ_{lgw} B$. By hypothesis of this case $A=B$ which is a contradiction. Then rank$(T(A))=1$.

Case 2. Suppose that $|\mathscr{R}(A)|>1$. Then by part (ii) of Proposition 3.1, $W=\{B\in\mathbf{M}_{n,m} : A\succ_{lgw} B\}$ is a subspace of $\mathbf{M}_{n,m}$ with dimension n. So $T(W)=\{T(B) : A\succ_{lgw} B\}=$

$\{T(B) : T(A) \succ_{lgw} T(B)\}$ is a subspace of $\mathbf{M}_{n,m}$ with dimension n. Therefore by part (ii) of Proposition 3.1, $\text{rank}(T(A)) = 1$ and $|\mathscr{R}(T(A))| > 1$. ∎

The following theorem characterizes all rank-k-preservers on $\mathbf{M}_{n,m}(\mathbb{C})$.

Theorem 3.1. [6, Theorem 3] *If T is a rank-k-preserver on $\mathbf{M}_{n,m} := \mathbf{M}_{n,m}(\mathbb{C})$, then there exist invertible matrices $U \in \mathbf{M}_n$ and $V \in \mathbf{M}_m$ such that either*

$$T(X) = UXV,\ \forall X \in \mathbf{M}_{n,m},$$

or

$$m = n \text{ and } T(X) = UX^tV,\ \forall X \in \mathbf{M}_{n,m},$$

where A^t denotes the transpose of A.

Lemma 3.1. [5, Lemma 2.4] *Let A and B be two invertible matrices. Then the linear operator $T : \boldsymbol{M}_n \to \boldsymbol{M}_n$ defined by $T(X) = AX^tB$ for all $X \in \boldsymbol{M}_n$, is not a strong linear preserver of $\succ_{lgw}$.*

Now, we are ready to give another proof of Theorem 2.2, in the case $\mathbb{F} = \mathbb{C}$ by using the structure of rank-1-preservers.

Theorem 3.2. *Let $T : \mathbf{M}_{n,m} \to \mathbf{M}_{n,m}$ be a linear operator. Then T strongly preserves $\succ_{lgw}$ if and only if there exist invertible matrices $A \in \mathbf{GR}_n$ and $B \in \mathbf{M}_m$ such that $T(X) = AXB$ for all $X \in \mathbf{M}_{n,m}$.*

Proof. Assume that T strongly preserves $\succ_{lgw}$. By Proposition 3.2, Theorem 3.1 and Lemma 3.1, there exist invertible matrices $U \in \mathbf{M}_n$ and $V \in \mathbf{M}_m$ such that $T(X) = UXV$ for all $X \in \mathbf{M}_{n,m}$. Since U is invertible, so there exists a unique $x_0 \in \mathbb{C}^n$ such that $Ux_0 = e$. Put $X = [x_0|0] \in \mathbf{M}_{n,m}$. It is clear that $X \succ_{lgw} RX$ for every $R \in \mathbf{GR}_n$. Then $T(X) \succ_{lgw} T(RX)$ and hence $U[x_0|0]V \succ_{lgw} UR[x_0|0]V$ for every $R \in \mathbf{GR}_n$. It follows that $e = Ux_0 \succ_{lgw} URx_0$ for every $R \in \mathbf{GR}_n$. Then $URx_0 = e$ and hence $Rx_0 = x_0$ for all $R \in \mathbf{GR}_n$. Therefore $x_0 = \lambda e$ for some $\lambda \in \mathbb{C}$. Put $A = \lambda U$ and $B = (1/\lambda)V$. It is clear that $A \in \mathbf{GR}_n$ and $T(X) = AXB$ for all $X \in \mathbf{M}_{n,m}$. ∎

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